MATH 3A 9/14/10

Note Title 9/14/2010

SPECIAL LIMITS

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\quad \lim_{x\to 0}\frac{1-\cos x}{x}=0,\quad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

where $e\approx 2.718\ldots$

Ex $\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\sin x=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\sin x$

$$=1\cdot\sin 0=1\cdot 0=0$$

$\lim_{x\to 0}\frac{5(1-\cos x)}{x}=5\cdot\lim_{x\to 0}\frac{1-\cos x}{x}=5\cdot 0=0$ $\left\{\begin{array}{l}\text{CONST MULT RULE}\\ \lim_{x\to c} b f(x)\\ =b\lim_{x\to c} f(x)\end{array}\right\}$

$$\lim_{x\to 0}\frac{2\tan^2 x}{x}=\lim_{x\to 0}\frac{2\left(\frac{\sin x}{\cos x}\right)^2}{x}=2\lim_{x\to 0}\frac{\sin^2 x}{x}\cdot\frac{1}{\cos^2 x}$$

$$=2\lim_{x\to 0}\left(\sin x\cdot\frac{\sin x}{x}\cdot\frac{1}{\cos^2 x}\right)=2\left(\lim_{x\to 0}\sin x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos^2 x}\right)$$

$$=2\cdot 0\cdot 1\cdot 1=0$$

Ex: $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{\left(1-(\sin x)/\cos x\right)}{(\sin x-\cos x)}\frac{\cos x}{\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{\cancel{(\cos x-\sin x)}}{-\cancel{(\cos x-\sin x)}\cos x}$

$\lim_{x\to\frac{\pi}{4}}\frac{1}{-\cos x}=\frac{1}{-1/\sqrt{2}}=-\sqrt{2}$ $\left\{=-\frac{1}{\frac{\sqrt{2}}{2}}=-\frac{2}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}=-\frac{\cancel{2}\sqrt{2}}{\cancel{2}}=-\sqrt{2}\right.$

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 6x}=\lim_{x\to 0}\frac{\left(\frac{\sin(5x)}{(5x)}\right)5(x)}{\left(\frac{\sin(6x)}{(6x)}\right)6(x)}=\frac{5}{6}\lim_{x\to 0}\frac{\frac{\sin(5x)}{5x}}{\frac{\sin(6x)}{6x}}=\frac{5}{6}\frac{\lim_{x\to 0}\frac{\sin(5x)}{5x}}{\lim_{x\to 0}\frac{\sin(6x)}{6x}}$$

$$=\frac{5}{6}\cdot\frac{1}{1}=\frac{5}{6}$$

$\frac{x}{x}=1$

$$\boxed{\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e}$$

$e \approx 2.718\ldots$ check $(1.001)^{\frac{1}{.001}} \approx e$

$x = .001$

"ADVANCED" LIMITS $\lim_{x\to 1} \frac{\ln x}{x-1}$ hints: $\ln X^N = N \cdot \ln X$

$$\Rightarrow \lim_{x\to 1}\left(\frac{1}{x-1}\right)\ln x = \lim_{x\to 1} \ln\left[x^{\frac{1}{x-1}}\right]$$

$u = x-1$ → as $x \to 1$

$x = u+1$ $u \to 0$

$$= \lim_{u\to 0} \ln\left[(1+u)^{\frac{1}{u}}\right]$$

aside $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$

$$= \ln\left[\lim_{u\to 0}(1+u)^{\frac{1}{u}}\right] = \ln[e] = 1 \Rightarrow \lim_{x\to 1}\frac{\ln x}{x-1} = 1$$

Ex $\lim_{x\to \ln 2} \frac{e^{3x}-8}{e^{2x}-4}$

numerator: $e^{3\ln 2} - 8 = e^{\ln(2^3)} - 8$

$= 2^3 - 8 = 8-8 = 0$

$$\lim_{x\to \ln 2} \frac{(e^x)^3 - (2)^3}{(e^x)^2 - (2)^2}$$

similarly: denom $= 0$

$a^2 - b^2 = (a-b)(a+b)$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$\lim_{x\to \ln 2} \frac{\cancel{(e^x-2)}(e^{2x}+2e^x+4)}{\cancel{(e^x-2)}(e^x+2)}$$

$$\lim_{x\to \ln 2} \frac{e^{2x}+2e^x+4}{e^x+2} = \frac{e^{2\ln 2} + 2e^{\ln 2} + 4}{e^{\ln 2}+2} = \frac{4+(2)(2)+4}{2+2}$$

$2\ln 2 \to \ln 4 \to \ln(2^2)$; $e^{\ln 2} = 2$

$= 3$

ASIDE $e^{2\ln 2} = e^{\ln(2^2)} = 2^2 = 4$

$$\lim_{x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$ SYMBOL to denote this limit is $f'(x)$

① $f(x)=\sqrt{x}$ find $f'(x)$

cross terms

$$\lim_{\Delta x\to 0}\frac{(\sqrt{x+\Delta x}-\sqrt{x})}{\Delta x}\cdot\frac{(\sqrt{x+\Delta x}+\sqrt{x})}{(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{x+\Delta x-x+0}{\Delta x(\sqrt{x}+\sqrt{x+\Delta x})}$$

$$\lim_{\Delta x\to 0}\frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})}=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x}+\sqrt{x+\Delta x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

② $\boxed{f(x)=\frac{1}{x+3}}$ find $f'(x)$ $\boxed{=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}}$

$$f(\ )=\frac{1}{(\ )+3}$$

$x+\Delta x$ $\quad$ $x+\Delta x$

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{1}{x+\Delta x+3}-\frac{1}{x+3}\right]$$

$$\boxed{f(x+\Delta x)=\frac{1}{x+\Delta x+3}}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ab-bc}{bd}\rightarrow\frac{1}{x+\Delta x+3}-\frac{1}{x+3}=\frac{x+3-(x+\Delta x+3)}{(x+3)(x+\Delta x+3)}$$

$$=\frac{x+3-x-\Delta x-3}{(x+3)(x+\Delta x+3)}=\frac{-\Delta x}{(x+3)(x+\Delta x+3)}$$

$$\Rightarrow\lim_{\Delta x\to 0}\frac{1}{\cancel{\Delta x}}\frac{-\cancel{\Delta x}}{(x+3)(x+\Delta x+3)}=\lim_{\Delta x\to 0}\frac{-1}{(x+3)(x+\Delta x+3)}$$

$$=\frac{-1}{(x+3)^2}\Rightarrow f(x)=\frac{1}{x+3},\ f'(x)=\frac{-1}{(x+3)^2}$$

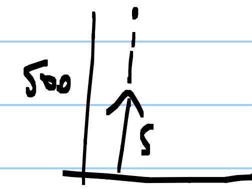
POSITION FUNCTION

$$s(t) = -16t^2 + h$$

describes the height of an object dropped from a height "h" units above ground at $t > 0$

Ex $h = 500$ $s(t) = -16t^2 + 500$

Suppose a "wrench" is dropped from a height of 500 ft.
How high above the ground is it after 2 s.



$$s(2) = -16(2)^2 + 500$$
$$= -16 \cdot 4 + 500 = -64 + 500 = 436 \text{ ft.}$$

How long does it take to reach ground $(s(t) = 0)$

$$0 = -16t^2 + 500 \rightarrow t = \sqrt{\frac{500}{16}} = \frac{\sqrt{10 \cdot 2 \cdot 25}}{4} = \frac{5 \cdot \cancel{2}\sqrt{5}}{2\cancel{4}} = \frac{5\sqrt{5}}{2}$$

$$\sqrt{4} < \sqrt{5} < \sqrt{9}$$
$$2 < \sqrt{5} < 3$$

$$\sqrt{5} \approx 2.2$$

$$\frac{5\sqrt{5}}{2} = \frac{5(2.2)}{2} = 5(1.1) = 5.5$$

given $S(t) = -16t^2 + h$

can find VELOCITY AT ANY TIME

By evaluating $V(a) = \lim_{t \to a} \frac{S(t) - S(a)}{t - a}$

speed $= \frac{\text{distance}}{\text{time}}$ a <u>FIXED number</u>   <u>t variable</u>

<u>Ex</u>: find for $S(t) = -16t^2 + 500$, the velocity when $t = \underbrace{2}_{\text{"}a\text{"}} \Rightarrow V(2) = \text{"}S'(2)\text{"}$

ALGEBRA FIRST

$$V(a) = \lim_{t \to a} \frac{S(t) - S(a)}{t-a} = \lim_{t \to a} \frac{(-16t^2 + 500) - (-16a^2 + 500)}{t - a}$$

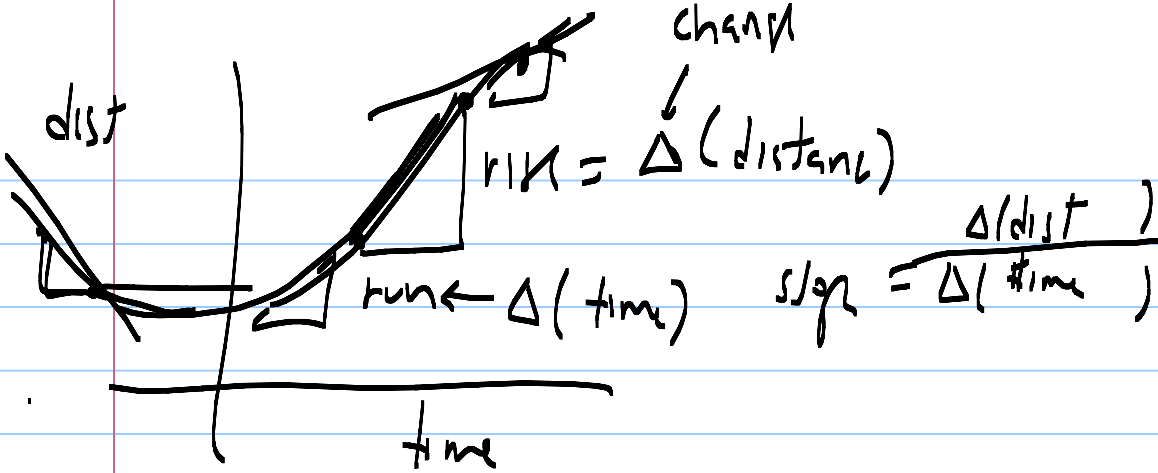
$$= \lim_{t \to a} \frac{-16(t^2 - a^2)}{t - a} = \lim_{t \to a} \frac{-16(t+a)\cancel{(t-a)}}{\cancel{t-a}} = \lim_{t \to a} -16(t + a)$$

$$= -16(2a)$$

$$= -32a$$

$\Rightarrow$ FOR THIS POSITION FUNCTION $S(t) = -16t^2 + 500$ we see that $V(a) = -32a$

$\Rightarrow t = \underbrace{2}_{a} \Rightarrow V(2) = -32(2) = \underbrace{-64 \text{ ft/s}}$ <u>down</u>



5th ed 2.4 CONTINUITY / 1 SIDED LIMITS

"DEFINITION"

f is continuous at $x = c$ if:

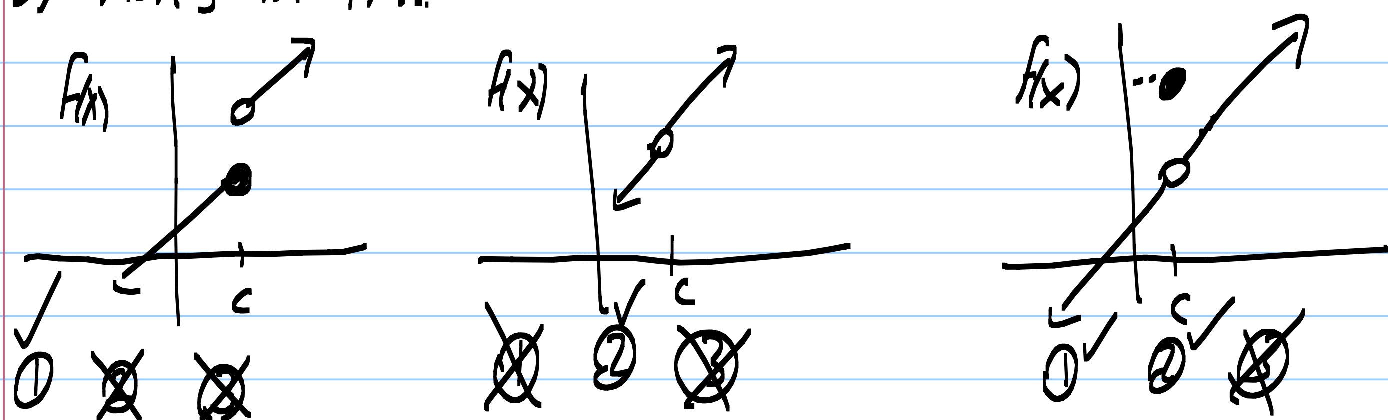
① $f(c)$ is defined

② $\lim_{x \to c} f(x)$ exists

③ $\lim_{x \to c} f(x) = f(c)$

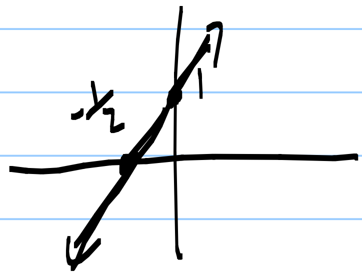
Definition makes sure that a graph has no breaks or holes in it!!!

Ex



INTUITIVE DEFN: A graph is continuous if it has no holes or breaks in it!!!

Ex $f(x) = 2x+1$



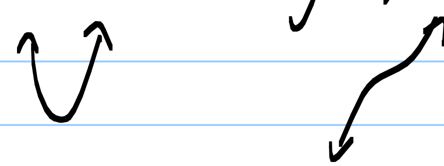
f is continuous over all real numbers

continuous over $\mathbb{R}$

continuous over $(-\infty, \infty)$ " "

Ex: FUNCTIONS THAT ARE CONTINUOUS OVER $\mathbb{R}$

POLYNOMIALS Ex $f(x) = x^2+1$ $g(x) = x^3$ etc



SINE / COSINE

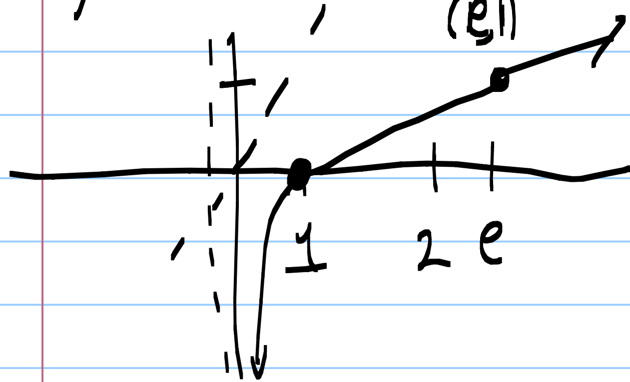
① $y = \sin x$

② $y = \cos x$

MANY FUNCTIONS ARE CONTINUOUS OVER A SUBSET OF $\mathbb{R}$

Ex $f(x) = \ln x$

$y = \ln x$ what interval is this continuous over ???



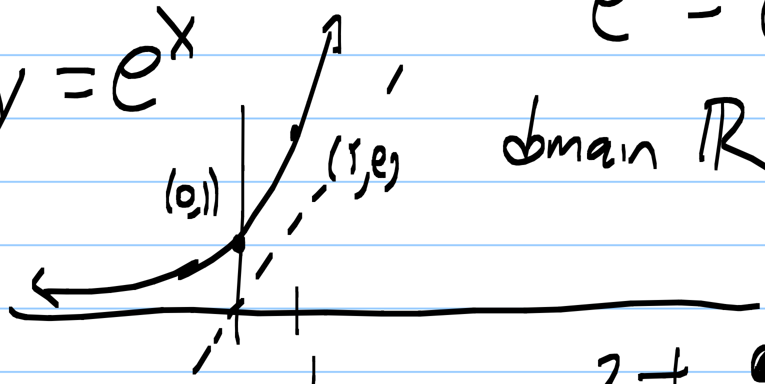
$(0, \infty)$ also this is the DOMAIN

$\ln(0) = ?$

$e^{?} = 0$

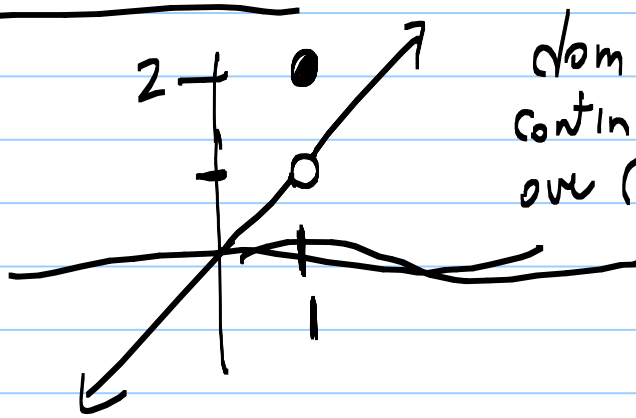
Compare this to.....

$y = e^x$



domain $\mathbb{R}$ continuous over $(-\infty, \infty)$

$$f(x) = \begin{cases} x & x \neq 1 \\ 2 & x = 1 \end{cases}$$

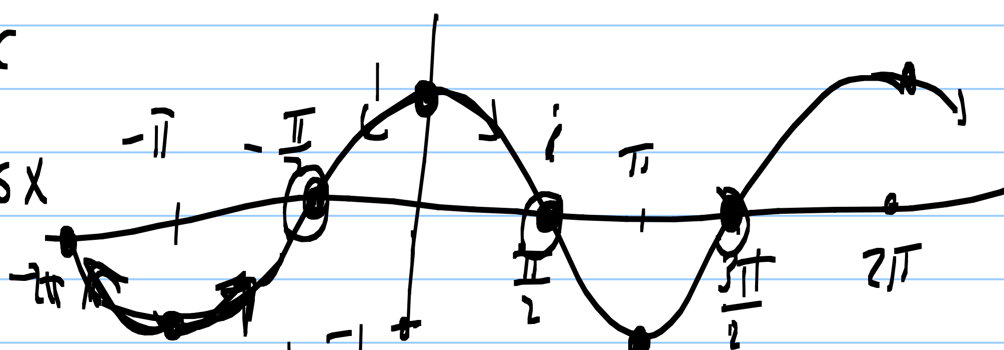


domain $\mathbb{R}$

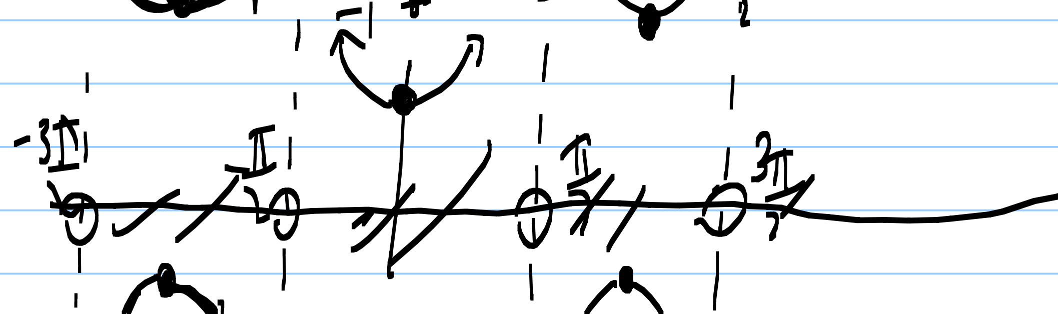
continuous over $(-\infty, 1) \cup (1, \infty)$

$f(x) = \sec x$

① $y = \cos x$



② $y = \frac{1}{\cos x}$



"BAD" POINTS $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots = \frac{(2n+1)\pi}{2}$

$n=0,-1$ $n=1,-2$, etc..

n is an integer

$y = \sec x$ is continuous over all real numbers except

$$\boxed{x = \frac{(2n+1)\pi}{2} \quad n \in \mathbb{Z}}$$

is an element

$\mathbb{Z}$ "integers"

$2n+1$ is how to represent ODD #

n is an integer

$2n$

GERMAN WORD FOR INTEGERS STARTS WITH $\mathbb{Z}$

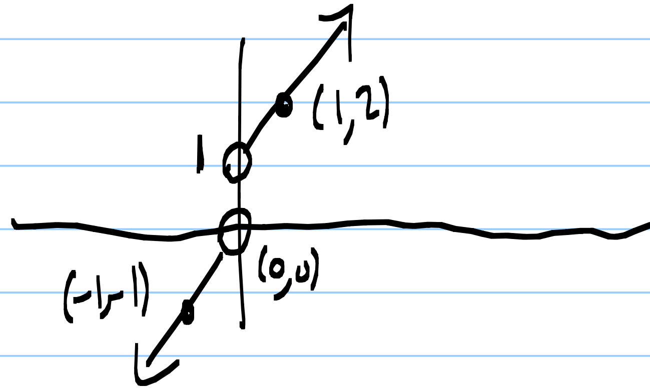
$y = \sec x$ is continuous over

$\ldots \cup \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots$

also write $\mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2}, n \in \mathbb{Z} \right\}$

ONE SIDED LIMITS

$$f(x) = \begin{cases} x & x < 0 \\ x+1 & x > 0 \end{cases}$$



$\lim_{x \to 0} f(x)$ DNE however consider

1-SIDED LIMITS

COMING FROM LEFT, LIMIT IS 0,

$\lim_{x \to 0^-} f(x) = 0$

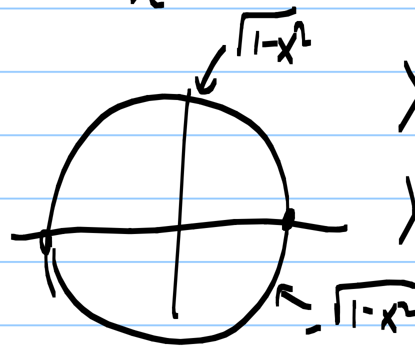
COMING from RIGHT, LIMIT IS 1

$\lim_{x \to 0^+} f(x) = 1$

FACT: A LIMIT EXISTS ONLY WHEN ONE-SIDED LIMITS AGREE: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

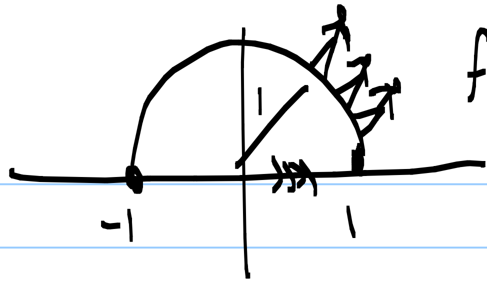
Ex $f(x) = \sqrt{1-x^2}$

$x^2 + y^2 = 1$



$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$



$f(x) = \sqrt{1-x^2}$ doman $[-1, 1]$

$\lim_{x \to 1} f(x)$ DNE b/c $\lim_{x \to 1^+} f(x)$ DNE

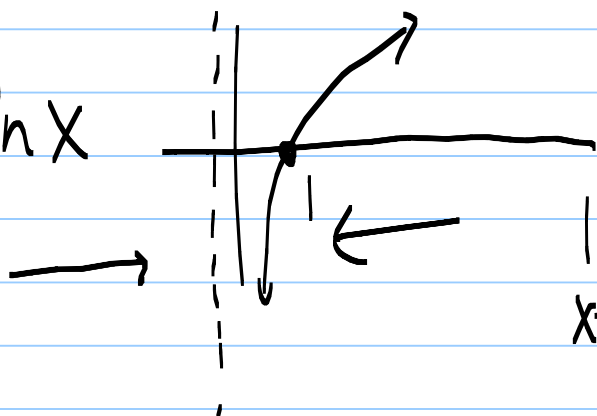
b/c outsid domain

$\lim_{x \to 1^-} f(x) = 0$

$\lim_{x \to 0} \ln x$ DNE !!!

why

$y = \ln x$



$\lim_{x \to 0^-} \ln x$ DNE outside domain

can considr

$\lim_{x \to 0^+} \ln x$ but it DNE b/c grph falls down to "$-\infty$"

$\lim_{x \to 0^+} \ln x = -\infty$ $\lim_{x \to 0^-} \ln x$ DNE DNE

NEXT TIME: ONE SIDED LIMITS

CONTINUED.